Session 9 : solutions

Exercise 1 :

$$\begin{cases} e^{2K_1'} = e^{2K_1} \dfrac{\cosh(K_1 + 2K_2)}{\cosh(K_1 - 2K_2)} \\[2ex] e^{4K_2'} = \dfrac{\cosh(K_1+2K_2)\cosh(K_1-2K_2)}{\cosh(K_1)^2} \end{cases}$$

$$\begin{cases} K_1' = K_1^* + \delta K_1' \\ K_2' = K_2^* + \delta K_2' \end{cases}$$

We work first on the first one :

$$e^{2K_1^*}(1 + 2\delta K_1') = e^{2K_1^*}(1 + 2\delta K_1) \cdot$$

$$\cdot \frac{\cosh(K_1^* + 2K_2^* + \delta K_1 + 2\delta K_2)}{\cosh(K_1^* - 2K_2^* + \delta K_1 - 2\delta K_2)} =$$

$$= e^{2K_1^*}(1 + 2\delta K_1) \frac{\cosh(K_1^* + 2K_2^*) + \sinh(K_1^* + 2K_2^*)(\delta K_1 + 2\delta K_2)}{\cosh(K_1^* - 2K_2^*) + \sinh(K_1^* - 2K_2^*)(\delta K_1 - 2\delta K_2)}$$

$$e^{2k_1^*}(1+2\delta k_1') =$$

$$= e^{2k_1^*}(1+2\delta k_1)\,\frac{\cosh(k_1^*+2k_2^*)}{\cosh(k_1^*-2k_2^*)}\cdot\frac{1+\mathrm{th}(k_1^*+2k_2^*)(\delta k_1+2\delta k_2)}{\underbrace{1+\mathrm{th}(k_1^*-2k_2^*)(\delta k_1-2\delta k_2)}_{\text{Taylor expand the denominator}}} =$$

$$= e^{2k_1^*}(1+2\delta k_1)\,\frac{\cosh(k_1^*+2k_2^*)}{\cosh(k_1^*-2k_2^*)}\left(1+\mathrm{th}(k_1^*+2k_2^*)(\delta k_1+2\delta k_2)\right)\cdot$$

$$\cdot\left(1-\mathrm{th}(k_1^*-2k_2^*)(\delta k_1-2\delta k_2)\right)$$

$$= e^{2k_1^*}\,\frac{\cosh(k_1^*+2k_2^*)}{\cosh(k_1^*-2k_2^*)}(1+2\delta k_1)\Big(1+\mathrm{th}(k_1^*+2k_2^*)(\delta k_1+2\delta k_2)+$$

$$\underbrace{-\,\mathrm{th}(k_1^*-2k_2^*)(\delta k_1-2\delta k_2)\Big)}_{\text{just keep linear order}}$$

$$= \underbrace{e^{2k_1^*}}_{\text{used the fixed point equation}}\Big[1+\left(2+\mathrm{th}(k_1^*+2k_2^*)-\mathrm{th}(k_1^*-2k_2^*)\right)\delta k_1 +$$

$$+2\left(\mathrm{th}(k_1^*+2k_2^*)+\mathrm{th}(k_1^*-2k_2^*)\right)\delta k_2\Big]$$

and at last

$$\delta k_1' = \left[1+\frac{1}{2}\left(\mathrm{th}(k_1^*+2k_2^*)-\mathrm{th}(k_1^*-2k_2^*)\right)\right]\delta k_1 +$$

$$+\left(\mathrm{th}(k_1^*+2k_2^*)+\mathrm{th}(k_1^*-2k_2^*)\right)\delta k_2$$

Now we work on the second equation in the same spirit

$$e^{4k_2'} = \frac{\cosh(k_1+2k_2)\cosh(k_1-2k_2)}{\cosh(k_1)^2}$$

$$e^{4k_2^*}(1+4\delta k_2') = \frac{\cosh(k_1^*+2k_2^*)\left[1+\mathrm{th}(k_1^*+2k_2^*)(\delta k_1+2\delta k_2)\right]\cdot \mathrm{ch}(k_1^*-2k_2^*)\cdot\left[1+\mathrm{th}(k_1^*-2k_2^*)(\delta k_1-2\delta k_2)\right]}{\cosh(k_1^*)^2 + \underbrace{2\cosh(k_1^*)\sinh(k_1^*)\,\delta k_1}_{\text{stop at first order}}}$$

$$\cancel{e^{4k_2^*}}(1+4\delta k_2') = \cancel{\frac{\cosh(k_1^*+2k_2^*)\,\mathrm{ch}(k_1^*-2k_2^*)}{\cosh(k_1^*)^2}}\;\frac{1+\mathrm{th}(k_1^*+2k_2^*)(\delta k_1+2\delta k_2)+\mathrm{th}(k_1^*-2k_2^*)(\delta k_1-2\delta k_2)}{1+2\,\mathrm{th}(k_1^*)\,\delta k_1}$$

$$1+4\delta k_2' = \left[1+\left(\mathrm{th}(k_1^*+2k_2^*)+\mathrm{th}(k_1^*-2k_2^*)\right)\delta k_1 + 2\left(\mathrm{th}(k_1^*+2k_2^*)-\mathrm{th}(k_1^*-2k_2^*)\right)\delta k_2 - 2\,\mathrm{th}(k_1^*)\,\delta k_1\right]$$

At last

$$\delta k_2' = \frac{1}{4}\left[\mathrm{th}(k_1^*+2k_2^*)+\mathrm{th}(k_1^*-2k_2^*)-2\,\mathrm{th}(k_1^*)\right]\delta k_1 + $$

$$+\frac{1}{2}\left(\mathrm{th}(k_1^*+2k_2^*)-\mathrm{th}(k_1^*-2k_2^*)\right)\delta k_2\,\Big]$$

The fixed points are:

1) $K_2^* = 0$ $\quad$ $K_1^* = \forall K_1$ $\quad$ $T = \infty$

2) $K_2^* = \infty$ $\quad$ $K_1^* = 0$ $\quad$ $T = 0$

makes sense: choose the field you want, $T=\infty$, it will not make any difference

Let's focus on fixed point 1:

$$\delta K_2' = 0 \cdot \delta K_2$$

$\longrightarrow$ again we should go to second order

$\Longrightarrow$ contraction

(since $K_1^* = 0$ it is exactly as before)

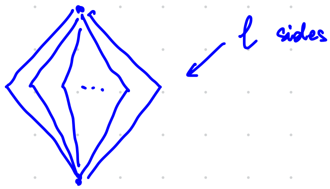
$$\delta K_1' = \left(1 + \tanh(2K_2^*) \right) \delta K_1 =$$

$$\underset{K_2^* = 0}{=} \delta K_1$$

this is an expansion

Exercise 2

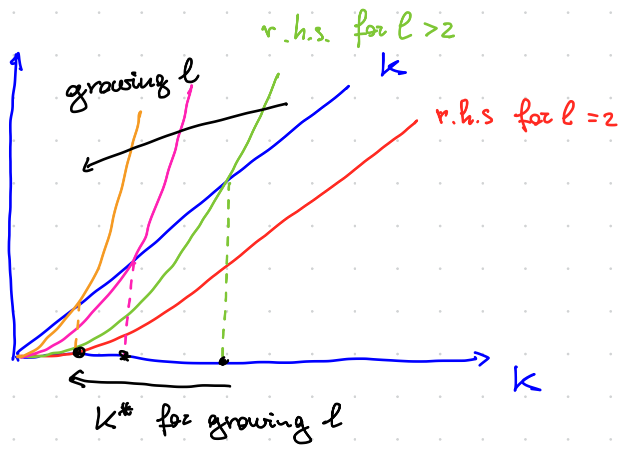
The construction is



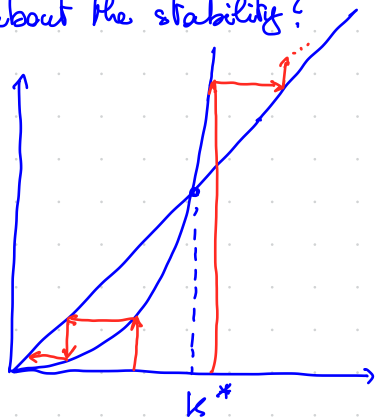
Then the coarse-grained iteration is

$$e^{4K^{(n+1)}} = \left[\cosh\left(2K^{(n)}\right)\right]^{\ell}$$

$$K^{(n+1)} = \frac{\ell}{4} \ln \cosh\left(2K^{(n)}\right)$$



What about the stability?



K^* is unstable $\rightarrow$ critical point.

What about y_t ?

$$\cancel{e^{4K^*}}\left(1+4\,\delta K'\right) = \left[\cosh(2K^*) + 2\,\mathrm{sh}(2K^*)\,\delta K\right]^{\ell} =$$

$$= \cancel{\cosh(2K^*)^{\ell}}\left[1 + \underbrace{2\ell\tanh(2K^*)\,\delta K}_{\text{first order}}\right]$$

They solve the fixed point

Then

$$4\,\delta K' = 2\ell\tanh(2K^*)\,\delta K$$

$$\Rightarrow \quad \delta K' = \frac{\ell}{2}\tanh(2K^*)\,\delta K$$

This means that

$$b^{y_t} = \frac{\ell}{2} \tanh(2K^*)$$

What can we say about K^* in the large ℓ limit?

$K^* \to 0$ and we can Taylor expand the fixed point equation

$$K^* \simeq \frac{\ell}{2} \ln\left[1 + \frac{1}{2} K^{*2}\right] \simeq \frac{\ell}{2} \frac{1}{2} K^{*2}$$

$$\Rightarrow \quad K^* = 0 \qquad (\text{fixed point})$$

$$K^* = \frac{4}{\ell} \qquad (\text{critical point})$$

At the critical point we have

$$b^{y_t} \simeq \frac{\ell}{2} \tanh\left(2 \cdot \frac{4}{\ell}\right) = \frac{\ell}{2} \tanh\Big(\underbrace{\frac{8}{\ell}}_{\ll 1 \text{ if } \ell \text{ big}}\Big) \simeq 4$$

The length factor $b = 2$

it is a strange generalisation of the 1D model

Then

$$2^{y_t} = 4 \longrightarrow \boxed{\lim_{\ell \to \infty} y_t = 2}$$

This is the mean-field result!